Spontaneous symmetry breaking: exact results for a biased random walk model of an exclusion process

Spontaneous symmetry breaking: exact results for a biased random walk model of an exclusion process

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Abstract. It has been recently suggested that a totally asymmetric exclusion process with two species on an open chain could exhibit spontaneous symmetry breaking in some range of the parameters defining its dynamics. The symmetry breaking is manifested by the existence of a phase in which the densities of the two species are not equal. In order to provide a more rigorous basis to these observations we consider the limit of the process when the rate at which particles leave the system goes to zero. In this limit the process reduces to a biased random walk in the positive quarter plane, with specific boundary conditions. The stationary probability measure of the position of the walker in the plane is shown to be concentrated around two symmetrically located points, one on each axis, corresponding to the fact that the system is typically in one of the two states of broken symmetry in the exclusion process. We compute the average time for the walker to traverse the quarter plane from one axis to the other, which corresponds to the average time separating two flips between states of broken symmetry in the exclusion process. This time is shown to diverge exponentially with the size of the chain.

1. Introduction

It is well known that systems in thermal equilibrium do not exhibit spontaneous symmetry breaking in one dimension at finite temperature, provided that the interactions are short range and that the local variable which describes the microscopic state of the system takes only a finite number of possible values [1]. Common examples of such systems are Ising or Potts-like models. On the other hand, when the local variable takes one of an infinite number of possible values the model may exhibit phase transitions and symmetry breaking in one dimension. Examples are solid-on-solid models which are used to describe wetting phenomena or unbinding transitions of a one-dimensional interface from an attractive wall [2].

Systems which are not in thermal equilibrium, but rather evolve under some stochastic dynamics, are different. In many cases these systems reach a steady state which does not obey detailed balance. The question is whether such a steady state of a one-dimensional system can break the symmetry of the dynamical rules under which the system evolves. A related problem has been considered in the context of error correcting computation

algorithms, and an example of a one-dimensional array of probabilistic cellular automata which exhibits non-ergodicity (as would a model with spontaneous symmetry breaking) has been constructed [3]. However, this example is rather complicated and not widely understood.

Recently a simple model was introduced [4, 5], describing a totally asymmetric exclusion process for two species of particles moving on an open chain of length N. Each lattice site may be occupied by either a positive (+) or a negative (−) particle, or by a hole (0). Positive particles move to the right, negative particles to the left. During an infinitesimal time interval dt the following exchange events may take place. On the one hand, positive particles can exchange with adjacent holes to their right, negative particles with adjacent holes to their left, and positive particles with adjacent negative particles to their right, at rates 1, 1, and q, respectively. That is, between two adjacent sites one has

$$+0 \to 0+ \qquad 0- \to -0 \qquad +- \to -+ \tag{1.1}$$

with probabilities dt, dt and q dt. On the other hand, positive particles are introduced at the left end with rate α and leave the system at the right end with rate β, and conversely for negative particles. Thus, at site 1

$$0 \to + \qquad - \to 0 \tag{1.2}$$

with probabilities α dt and β dt respectively; at site N

$$0 \to - \qquad + \to 0 \tag{1.3}$$

with probabilities α dt and β dt respectively. The dynamical rules of the model are symmetric under the simultaneous interchange of the sign of particles and of left and right, i.e. the dynamics of the positive particles moving to the right is the same as that of the negative particles moving to the left. Therefore, as long as this symmetry is not spontaneously broken, the densities and currents of the positive and negative particles are expected to be equal.

The model has been studied in [4, 5] both in a mean-field approximation and by simulation. The mean-field phase diagram exhibits a broken symmetry phase when the rate β is smaller than a critical value $\beta_c(\alpha, q)$; in this phase the currents and the densities of the positive and negative particles are unequal. Numerical simulations of the model give similar results: the system evolves to one of two possible long lived states. These two states are related by the symmetry of simultaneous sign of particles and left–right interchange. In one of them the current and density of positive particles are larger than those of the negative particles and, in the other, these inequalities are reversed. Outside of small regions near the boundaries, the states appear to be spatially uniform, and in the $N \to \infty$ limit they become true steady states, hereafter also referred to as *phases*, to be identified with those of the infinite system studied, in the case $q = 1$, in [6]. For any finite size N the system flips between these two possible states as it evolves, resulting in an overall symmetric state. This dynamics is characterized by a time scale $\langle T \rangle$, the average time between two consecutive flips, which diverges in the $N \to \infty$ limit. In [4, 5] it has been suggested, using numerical simulations and some simple considerations, that $\langle T \rangle$ grows exponentially with N. The flipping process itself takes place over a time interval much less than $\langle T \rangle$.

To summarize, the two-species exclusion process studied in [4, 5] is known, at least on a heuristic basis, to exhibit the following two properties (when $\beta < \beta_c$):

(i) the system spends most of its time in one of the two symmetry related long lived states described above, each corresponding, in the $N \to \infty$ limit, to a spatially homogeneous phase over the entire system;

(ii) the average time between two consecutive flips, i.e. the average lifetime of a long lived state, is exponentially increasing with the size of the system:

$$\langle T \rangle \sim \exp(N\mu). \tag{1.4}$$

The positive constant μ in (1.4), hereafter referred to as the *mass*, is the analogue in the present situation of the barrier height, or activation energy, per site, characteristic of the law of Arrhenius for systems at equilibrium.

These two properties, in analogy with the situation which prevails in the case of the two-dimensional Ising model below the critical temperature, lead us to say that the model exhibits spontaneous symmetry breaking. The analogy between the two models may be pursued. The escape rate β in the exclusion process plays the role of temperature, α and q being kept fixed. In particular, density fluctuations increase with β, and a transition from a broken symmetry phase to a symmetric phase is found when $\beta = \beta_c$. Finally changing the input and output rates at the boundaries to produce an asymmetry between positive and negative particles is akin to introducing a magnetic field in the Ising model (see the conclusion).

The aim of the present paper is to demonstrate the existence of spontaneous symmetry breaking in the $\beta \to 0$ limit of the two-species exclusion model, by showing that the two properties mentioned above hold. In this limit the model can be mapped onto a much simpler *toy model*, for which the set of available configurations is restricted to those composed of three blocks, with j negative particles on the left, v holes in the middle and k positive particles on the right, with $N = j + k + v$ (see equation (2.1)). The dynamics of these configurations may be represented by a biased random walk in the positive quarter plane (j, k), with boundary conditions to be defined below. We demonstrate that the toy model flips between two long lived states. These states coincide, except near the boundaries, with one or the other pure phase of the system, i.e. typically the larger of the $(-)$ and $(+)$ blocks fills the system except for a region whose size does not grow with N: with high probability, $\max(j, k) = N - O(1)$.

We are able to compute the average time between two consecutive flips, and find an exponential growth with the system size according to equation (1.4).

Let us finally emphasize that both properties:

(i) existence of pure phases (related by symmetry),

(ii) divergence of the time scale $\langle T \rangle$ with the system size,

are necessary in order to claim spontaneous symmetry breaking, neither of them taken separately being sufficient.

This may be illustrated by the following case. In the one-species exclusion process on an open chain, with input rate α and output rate β, studied in [7–10], the typical configuration on the phase coexistence line $0 < \alpha = \beta < \frac{1}{2}$ consists of two regions of size $O(N)$, one in each phase (one of low density α, the other of high density $1 - \alpha$), separated by a *shock*. The shock diffuses, hence its position is uniformly distributed over the chain in the steady state, and there is a characteristic time of order N^2 for the shock to traverse the system and hence for the system to change from one pure phase to another, without ever staying in one of them. In other words, there are no pure long lived states in this case; the steady state is a mixture. In conclusion, though the time scale $\langle T \rangle$ necessary for the system to pass from one phase to the other (without staying in one of them) diverges with the size,

no spontaneous symmetry breaking occurs in this case. Note that the N^2 divergence of $\langle T \rangle$, instead of an exponential one, comes together with the absence of pure phases. This situation is presumably generic. We will come back to this example in section 5 since it corresponds to a limiting case of the two-species process.

This paper is organized as follows. In section 2 we present simple physical arguments suggesting that, in the $\beta \to 0$ limit, the dynamics of the exclusion process may be mapped onto that of the toy model. This correspondence is made rigorous in appendix A. Section 3 provides a description of the toy model as a biased random walk in the positive quarter plane. The equations for the stationary probability are given in section 4, as well as those for the average time between two flips. The latter involves two intermediate functions $u_{j,k}$ and $v_{j,k}$, which have simple probabilistic interpretations. Appendix B provides a convenient way to compute all these quantities numerically. Section 5 is devoted to the study of two simple limiting cases of the random walk which are helpful to understand the general case. The analysis of the general case is given in section 6, where the asymptotic expressions of the stationary probability measure and of the average time between two flips are derived. We thus show that the steady state in the toy model is concentrated on configurations in which the system is almost entirely in a single phase. The analytic expression of the mass μ is given. In section 7 we determine the scaling behaviour of $u_{j,k}$, $v_{j,k}$ and the average time between two flips in the continuum limit, corresponding to a small bias and large distances. A summary and a discussion are given in section 8.

2. Definition of the toy model

In this section we show by means of simple physical arguments that in the limit where the rate β at which particles leave the system is small compared to the other rates defining the dynamics—α (particle input), q (particles interchange) and 1 (particle–hole exchange)—the exclusion model reviewed in the introduction may be mapped onto a simpler model, which we term the *toy model*. A rigorous proof of the equivalence of the two models is provided in appendix A.

The time scale for the toy model is determined by the escape rate β, and we therefore introduce a rescaled time $\tau = 2\beta t$. In the $\beta \to 0$ limit the only relevant configurations are those composed of three blocks, with negative particles to the left, holes in the middle and positive particles to the right: a typical configuration is of the form

$$\overbrace{\underbrace{- \cdots -}_{j} \, \underbrace{0 \cdots 0}_{\nu} \, \underbrace{+ \cdots +}_{k}}^{N}. \tag{2.1}$$

All other configurations may be neglected. This can be understood as follows.

First, since β is small, outputs of particles are rare on the time scale of the exclusion model. Any configuration different from those above will therefore rearrange itself— quickly, on the τ time scale—until all the particles are waiting to exit. Suppose now that a particle exits. Either it will be replaced by a particle of the opposite sign, which will quickly traverse the system until it reaches the block of particles of its sign, or the hole thus created will itself travel through the system to join the block of holes in the middle. On the τ time scale, then, the dynamics of the system involves transitions between various three-block configurations. These transitions may be described only in terms of the events taking place on the boundaries. Note that if the particle which exits is the last of its type,

then the system will quickly fill with particles of the other type, leaving a configuration with only one block. Moreover, should the system empty completely, particles will quickly enter and rearrange to the form (2.1). Thus all configurations with $j = 0$ or $k = 0$, except for those with $k = N$ or $j = N$, respectively, may be neglected in the $\beta \to 0$ limit.

Let us now give a quantitative description of the dynamics in this configuration space. We denote by (j, k) a three-block configuration, according to equation (2.1). The integers j and k may take the values $0, 1, \ldots, N$, with the restriction that $j + k \leqslant N$. Configurations $(0, k)$ with $0 \leqslant k < N$ and $(j, 0)$ with $0 \leqslant j < N$ do not occur.

Consider now what may happen during a time interval $\Delta\tau$ which is short on the τ time scale but long on the t scale: $\beta \ll \Delta\tau \ll 1$. We can ignore the possibility that more than one particle exits in such an interval. If $j \geqslant 1$, then there is a small probability $\frac{1}{2}\Delta\tau = \beta\Delta t$ that the negative particle at site 1 leaves the system during this time interval. Suppose that this happens.

(i) If $j > 1$, then immediately (on the τ scale) either the hole thus created is filled by a positive particle, or the negative particle on site 2 exchanges position with the hole; the relative probabilities of these events are $\alpha/(1 + \alpha)$ and $1/(1 + \alpha)$. Either event will be followed by a fast reordering of the particles, resulting in the configuration $(j - 1, k + 1)$ or $(j - 1, k)$, respectively, unless $j = N$, when in the second case an additional negative particle will enter once the hole reaches the right end of the interval, resulting in the final configuration $(N, 0)$, that is, in no net change.

(ii) If $j = 1$, then when the negative particle at site 1 leaves the system, the negative block disappears altogether. In this case positive particles are quickly introduced into the system at the left end and the system becomes filled with positive particles. The resulting configuration is $(0, N)$.

Similar processes take place at the right end when a positive particle leaves the system. To summarize these results we introduce rescaled rates

$$a = \frac{1}{2(1 + \alpha)} \qquad b = \frac{\alpha}{2(1 + \alpha)} \qquad \qquad (2.2)$$

which satisfy

$$a + b = \tfrac{1}{2}. \qquad (2.3)$$

Then during the time interval $d\tau$ the following stochastic changes take place in the system:

$$\begin{cases} (j, k) \to (j - 1, k + 1) & \text{if } j > 1, \text{ with probability } b\,d\tau \\ (j, k) \to (j - 1, k) & \text{if } N > j > 1, \text{ with probability } a\,d\tau \\ (1, k) \to (0, N) & \text{with probability } \tfrac{1}{2}\,d\tau \\ (j, k) \to (j + 1, k - 1) & \text{if } k > 1, \text{ with probability } b\,d\tau \\ (j, k) \to (j, k - 1) & \text{if } N > k > 1, \text{ with probability } a\,d\tau \\ (j, 1) \to (N, 0) & \text{with probability } \tfrac{1}{2}\,d\tau. \end{cases} \qquad (2.4)$$

Let us emphasize again that states $(j, 0)$ with $j < N$ and $(0, k)$ with $k < N$ do not occur.

These rules completely specify the toy model. They describe a random walk in the (j, k) plane, biased towards the south-west direction, that we study hereafter.

3. The toy model as a random walk

The rest of this paper is devoted to a quantitative analysis of the random walk defined by the dynamical rules (2.4) of the toy model. For the sake of clarity we summarize its definition first.

A random walker hops from site to site on a two-dimensional lattice, restricted to the positive quarter plane. The position of the walker is denoted by (j, k). During the time interval $d\tau$ it hops from (j, k) to the south $(j, k - 1)$ or to the west $(j - 1, k)$, each with probability $a\, d\tau$. It may also hop to the north-west $(j - 1, k + 1)$ or to the south-east $(j + 1, k - 1)$ each with probability $b\, d\tau$ $(a + b = \frac{1}{2})$. When the walker reaches the j or the k axis it restarts instantly at site $(N, 0)$ or $(0, N)$, respectively. The random walker is thus subjected to non-local boundary conditions. For short we will say that the axes are *bouncing* boundaries.

Let us determine the velocity vector and the diffusion tensor of this random walk. Consider the probability $p_{j,k}(\tau)$ to find the walker on site (j, k) at time τ. Its evolution in time is given by the following master equation, obtained by conditioning on the last step of the walker

$$\frac{d}{d\tau} p_{j,k}(\tau) = a(p_{j+1,k}(\tau) + p_{j,k+1}(\tau)) + b(p_{j+1,k-1}(\tau) + p_{j-1,k+1}(\tau)) - p_{j,k}(\tau) \qquad (3.1)$$

leaving aside boundary conditions. This equation may be rewritten as

$$\frac{d}{d\tau} p(x, \tau) = \sum_i a_i [p(x - e_i, \tau) - p(x, \tau)] \qquad (3.2)$$

with $x = (j, k)$, and where the e_i $(i = 1, \ldots, 4)$ correspond to the four possible moves. To the south: $e_1 = (0, -1)$, to the west: $e_2 = (-1, 0)$, to the north-west: $e_3 = (-1, 1)$, to the south-east: $e_4 = (1, -1)$, with rates $a_1 = a_2 = a$, and $a_3 = a_4 = b$ respectively.

The velocity vector reads [11, 12]

$$V = \lim_{\tau \to \infty} \frac{1}{\tau} \langle x(\tau) \rangle = \sum_i a_i e_i \qquad (3.3)$$

while the components of the diffusion tensor D are given by

$$D_{\mu\nu} = \lim_{\tau \to \infty} \frac{1}{2\tau} \{ \langle x_\mu(\tau) x_\nu(\tau) \rangle - \langle x_\mu(\tau) \rangle \langle x_\nu(\tau) \rangle \} = \frac{1}{2} \sum_i a_i (e_i)_\mu (e_i)_\nu. \qquad (3.4)$$

In the present case one finds

$$V = \begin{pmatrix} -a \\ -a \end{pmatrix} \qquad (3.5)$$

and

$$D = \frac{1}{2} \begin{pmatrix} 1 - a & 2a - 1 \\ 2a - 1 & 1 - a \end{pmatrix}. \qquad (3.6)$$

The eigenvalues of this matrix yield the two diffusion coefficients

$$D_\perp = \frac{2 - 3a}{2} \qquad D_\parallel = \frac{a}{2} \qquad (3.7)$$

where perpendicular and parallel refer to the south-west direction of the velocity (or bias). Note that the velocity and the parallel diffusion coefficient vanish as $a \to 0$.

The toy model captures the essence of the original exclusion process, when β is small; in particular the mechanism leading to spontaneous symmetry breaking has an intuitive explanation in terms of the random walk. Let us first review this mechanism for the exclusion process. Suppose the system is in the state of high density of (+) particles. In this phase there is a low flux of negative particles and holes entering the system at the right end and leaving at the left end. As the system evolves, a block of negative particles may *temporarily be formed at the left end due to some fluctuation.* As a consequence holes entering the system become trapped between the positive and negative regions. Usually, the block at the left end leaves the system after some time and the system relaxes to the all (+) state. However, if the block persists long enough, the system may be filled with holes, and thus has a chance of flipping to the all (−) state.

This mechanism has also a clear interpretation in the framework of the biased random walk of the toy model. Consider the walker starting at $(0, N)$, i.e. in the all (+) state. With large probability the walker will not go far east against the bias. It will therefore soon hit the k axis and bounce to its starting point $(0, N)$. However, with a small probability, vanishing as $N \to \infty$, it may reach the other end of the system $(N, 0)$. It is thus clear that the walker spends most of its time moving near one of the two endpoints of the system, occasionally traversing to the other end. Therefore the stationary probability measure of the walker is concentrated around both endpoints $(0, N)$ and $(N, 0)$. *The average time for the walker to move from one end of the system to the other is the average time between two flips in the toy model.* This time is denoted hereafter by $\langle T_0 \rangle$. It is related to the average time between two flips of the original exclusion process in the $\beta \to 0$ limit by

$$\langle T \rangle \approx \frac{\langle T_0 \rangle}{2\beta} \qquad (\beta \to 0). \tag{3.8}$$

The stationary probability measure as well as the average time between two flips in the toy model are studied in the next sections.

4. Equations for the stationary probability and for the average time between two flips

Two questions are to be answered:

(i) What is the probability $p_{j,k}$ for the walker to be at site (j, k) in the steady state (i.e. the stationary probability measure)?

(ii) What is the average time $D_{j,k}$ for the walker, starting from site (j, k), to reach any site on the j axis for the first time? The average time between two flips in the toy model is simply

$$\langle T_0 \rangle = D_{0,N}. \tag{4.1}$$

In this section we establish the equations fulfilled by these two quantities. Their solutions, mostly in the asymptotic $N \to \infty$ regime of interest, are presented in the next sections.

4.1. The stationary probability measure

In the steady state, the equation for the probability $p_{j,k}$ to find the walker on site (j, k) is

$$p_{j,k} = a(p_{j,k+1} + p_{j+1,k}) + b(p_{j+1,k-1} + p_{j-1,k+1}) \tag{4.2}$$

as a simple consequence of equation (3.1), with boundary conditions

$$
\begin{align}
p_{j,0} &= 0 \qquad (j \neq N) \notag \\
p_{0,k} &= 0 \qquad (k \neq N) \tag{4.3} \\
p_{j,k} &= 0 \qquad (j + k > N).
\end{align}
$$

The first two conditions are the only relevant consequences of the bouncing property of the axes.

The stationary probability measure obeys the symmetry property

$$p_{j,k} = p_{k,j}. \tag{4.4}$$

In addition the following normalization condition is imposed:

$$\sum_{j,k} p_{j,k} = 1. \tag{4.5}$$

4.2. The average time between two flips

$D_{j,k}$, as defined above, is the first-passage time through the j axis. It is therefore convenient to consider this axis as absorbing (the k axis is still bouncing). Conditioning on the first step of the walker, during the infinitesimal time interval $d\tau$, yields

$$
\begin{align}
D_{j,k} = a\,d\tau(D_{j,k-1} + d\tau) &+ a\,d\tau(D_{j-1,k} + d\tau) + b\,d\tau(D_{j+1,k-1} + d\tau) \notag \\
&+ b\,d\tau(D_{j-1,k+1} + d\tau) + [1 - (2a + 2b)\,d\tau](D_{j,k} + d\tau) \tag{4.6}
\end{align}
$$

hence

$$D_{j,k} = 1 + a(D_{j,k-1} + D_{j-1,k}) + b(D_{j+1,k-1} + D_{j-1,k+1}) \tag{4.7}$$

where j and $k = 1, \ldots, N$. The boundary conditions read

$$D_{0,k} = D_{0,N} \tag{4.8a}$$

$$D_{j,0} = 0 \tag{4.8b}$$

$$bD_{0,N} = bD_{1,N-1} + 1. \tag{4.8c}$$

The first equation reflects the non-local bouncing property of the k axis, while the second one expresses that the j axis is absorbing. The last equation comes from the fact that, starting from site $(0, N)$, the walker has probability $b\,d\tau$ to hop to site $(1, N - 1)$, and the complementary probability $1 - b\,d\tau$ to stay on this site. Note that the recursion relations found for the $D_{j,k}$, equations (4.7), are the same as those for a discrete-time process [13], since the hopping rates of the walker add to 1.

In order to unravel the cumbersome condition (4.8a) imposed by the bouncing boundary, it is convenient to introduce two functions $u_{j,k}$ and $v_{j,k}$ satisfying

$$u_{j,k} = 1 + a(u_{j,k-1} + u_{j-1,k}) + b(u_{j+1,k-1} + u_{j-1,k+1}) \qquad (4.9a)$$

$$v_{j,k} = a(v_{j,k-1} + v_{j-1,k}) + b(v_{j+1,k-1} + v_{j-1,k+1}) \qquad (4.9b)$$

with the boundary conditions

$$u_{0,k} = 0 \qquad v_{0,k} = 1 \qquad\qquad (4.10a)$$

$$u_{j,0} = 0 \qquad v_{j,0} = 0. \qquad\qquad (4.10b)$$

$u_{j,k}$ and $v_{j,k}$ obey the symmetry properties

$$u_{j,k} = u_{k,j} \qquad v_{j,k} + v_{k,j} = 1. \qquad\qquad (4.11)$$

Then the solution to equations (4.7) and (4.8a–c) is given by

$$D_{j,k} = u_{j,k} + D_{0,N} v_{j,k} \qquad (4.12)$$

and, in particular, by

$$D_{0,N} = \frac{u_{1,N-1} + 1/b}{1 - v_{1,N-1}}. \qquad (4.13)$$

We are actually interested in determining $D_{0,N}$ in the asymptotic regime, physically relevant, where N is large. As shown by equation (4.13) two independent sets of equations for $u_{j,k}$ and $v_{j,k}$ have to be solved separately. This problem is studied in the next sections.

We conclude by giving simple interpretations to the quantities met in this section and to the relations between them, in particular to the key equation (4.13).

(i) As can be deduced by considering the equations fulfilled by $u_{j,k}$, this quantity represents the average time for the walker starting from (j, k) to reach either of the axes for the first time. Similarly $v_{j,k}$ is the probability for the walker starting from (j, k) to reach the k axis without hitting the j axis. Generalizing the language of the classical *gambler's ruin* problem, $u_{j,k}$ is the *duration of the game*, and $v_{j,k}$ the *probability of ruin*: the walker is ruined when reaching the k axis, whereas it wins when reaching the j axis. The game stops in both cases. When considering the quantities $u_{j,k}$ and $v_{j,k}$ it is therefore convenient to consider both axes as absorbing.

Hence equation (4.13) also gets a simple interpretation. Its denominator is the probability for the walker, starting from $(1, N-1)$ to reach the j axis, that we may name the *probability of success*. In the numerator, $1/b$ is the average time for the walker starting from $(0, N)$ to reach site $(1, N - 1)$, while $u_{1,N-1}$ is the average time, starting from $(1, N - 1)$, to reach either of the axes. Altogether the numerator of (4.13) represents the average duration of an *attempt* for the walker starting from $(0, N)$: once the walker has reached site $(1, N - 1)$, either it succeeds with probability $1 - v_{1,N-1}$ in traversing the quarter plane, or it fails with probability $v_{1,N-1}$, returning to its starting point (see also equation (4.19) below). Alternatively this duration represents the average time between two *bounces* on either of the axes. Denoting this duration by $\langle T_b \rangle$, we have

$$\langle T_b \rangle = \frac{1}{b} + u_{1,N-1} \qquad (4.14)$$

and (4.13) reads, using (4.1),

$$\langle T_0 \rangle = \frac{\langle T_b \rangle}{1 - v_{1,N-1}}. \tag{4.15}$$

The probability of success is therefore the ratio of the frequency of flips (or success) to the frequency of bounces (or attempts).

(ii) The average time between two bounces $\langle T_b \rangle$ can be independently evaluated as follows. Just after a bounce the walker either passes through the bond from $(0, N)$ to $(1, N-1)$, or through the bond from $(N, 0)$ to $(N-1, 1)$. Hence $\langle T_b \rangle = 1/(2J)$, where $J = b p_{0,N}$ is the probability current through either of the two bonds. Thus

$$\langle T_b \rangle = \frac{1}{2b p_{0,N}}. \tag{4.16}$$

Comparing equations (4.14) and (4.16) yields

$$\frac{1}{2p_{0,N}} = 1 + b u_{1,N-1} \tag{4.17}$$

which will be used hereafter.

(iii) Finally, an alternative interpretation of (4.15) is provided by the following considerations. The average number of attempts of the walker until the first success is

$$\langle m \rangle = \sum_{m \geqslant 0} m (v_{1,N-1})^m (1 - v_{1,N-1}) = \frac{v_{1,N-1}}{1 - v_{1,N-1}}. \tag{4.18}$$

Denote by $u_{1,N-1}^{(j)}$ the average time for a walker starting from $(1, N-1)$ to touch the j axis without touching the k axis; $u_{1,N-1}^{(k)}$ is defined similarly, by exchanging the roles of j and k. Then

$$\langle T_0 \rangle = \langle m \rangle \left(\frac{1}{b} + u_{1,N-1}^{(k)} \right) + \left(\frac{1}{b} + u_{1,N-1}^{(j)} \right) \tag{4.19}$$

where the first parenthesis represents the average duration of a failure, the walker starting from $(0, N)$ and touching the k axis first, while the second one represents the average duration of a flip (or success), the walker starting from $(0, N)$ and touching the j axis first. Combining equations (4.18), (4.19) and noting that

$$\langle T_b \rangle = \frac{1}{b} + u_{1,N-1} = v_{1,N-1} \left(\frac{1}{b} + u_{1,N-1}^{(k)} \right) + (1 - v_{1,N-1}) \left(\frac{1}{b} + u_{1,N-1}^{(j)} \right) \tag{4.20}$$

again yields the expression (4.15) of $\langle T_0 \rangle$.

5. Two limiting cases

In this section we consider two simple limiting cases $a = 0$ and $b = 0$, which allow for exact closed-form solutions, that will be helpful to understand the general case. We use here the discrete-time language, for simplicity, since it yields the same equations and results as the continuous-time formalism used so far, because the hopping rates of the walker add to 1.

5.1. The case $a = 0$

In this limiting case the toy model reduces to the classical *gambler's ruin* problem on the segment $j + k = N$, with equal probabilities $b = \frac{1}{2}$ for the walker to hop to the right or to the left, and with reflections on sites $(0, N)$ and $(N, 0)$. The stationary probability measure is uniform on that segment, i.e.

$$p_{j,k} = \frac{1}{N+1} \qquad (j + k = N). \tag{5.1}$$

On the other hand, equations (4.10a) and (4.10b) have the following simple solutions:

$$u_{j,k} = jk \ . \qquad v_{j,k} = \frac{k}{j+k}. \tag{5.2}$$

By inserting these results into equations (4.13) and (4.14), we are left with

$$\langle T_b \rangle = N + 1 \qquad \langle T_0 \rangle = D_{0,N} = N(N+1). \tag{5.3}$$

The asymptotic N^2 law is due to the diffusive nature of the process. One also finds

$$D_{j,k} = N(N+1) - j(j+1). \tag{5.4}$$

5.2. The case $b = 0$

This case also yields a classical problem, '*Le problème des points*', considered by Fermat and Pascal [14].

In this case, once the random walker has reached either of the axes, it stays there. Therefore for $b = 0$ the model is not ergodic even for a finite system size, and any probability measure supported on the two points $(0, N)$ and $(N, 0)$, i.e. satisfying $p_{0,N} + p_{N,0} = 1$, is stationary. In the $b \to 0$ limit of the toy model we obtain the symmetric measure, i.e. $p_{0,N} = p_{N,0} = \frac{1}{2}$.

Let us now compute $u_{j,k}$ and $v_{j,k}$. Consider the more general case where the probability of hopping south is p and of hopping west is r, with $p + r = 1$. Equations (4.10) and (4.11) can be solved by means of generating functions. Dealing first with the $v_{j,k}$, we introduce the two-variable generating series

$$V(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} v_{j,k} x^j y^k. \tag{5.5}$$

Equation (4.10b) yields a closed equation for the function $V(x, y)$, which can be solved explicitly. We thus obtain the rational form

$$V(x, y) = \frac{rxy}{(1-y)(1-rx-py)}. \tag{5.6}$$

By expanding this expression as a double series, we obtain

$$v_{j,k} = \sum_{n=0}^{k-1} \binom{j+n-1}{n} p^n r^j. \tag{5.7}$$

Note that the nth term in the sum represents the probability that the walker, starting at (j, k), reaches the k axis after having made $n < k$ steps south. In particular one gets

$$v_{1,k} = 1 - p^k \qquad v_{j,1} = r^j. \tag{5.8}$$

In a similar way, equation (4.10a) yields the following rational expression for the generating series of the $u_{j,k}$

$$U(x, y) = \frac{xy}{(1 - x)(1 - y)(1 - rx - py)}. \tag{5.9}$$

By expanding this result we get

$$u_{j,k} = \sum_{m=0}^{j-1} \sum_{n=0}^{k-1} \binom{m+n}{m} r^m p^n \tag{5.10}$$

and especially

$$u_{1,k} = \frac{1 - p^k}{r} \qquad u_{j,1} = \frac{1 - r^j}{p}. \tag{5.11}$$

$\langle T_b \rangle$ and $\langle T_0 \rangle$ are divergent when $b \to 0$. Indeed by inserting the above results for $p = r = 1/2$ into equations (4.13), (4.14), we obtain their asymptotic behaviour in the $b \to 0$ regime:

$$\langle T_b \rangle \approx \frac{1}{b} \qquad \langle T_0 \rangle \approx \frac{2^{N-1}}{b} \qquad (b \to 0). \tag{5.12}$$

The second result has the form of equation (1.4), with a mass $\mu = \ln 2$.

Coming back to the original two-species exclusion process, one notes that the two cases considered here correspond to two extreme situations.

The case $a = 0$ corresponds to α infinite. In this case the stationary state is known [5]. No holes are present in the system and the remaining (+) and (−) particles play the role of particles and holes, respectively, in the equivalent one-species totally asymmetric exclusion process to which the original process reduces. Moreover, in this equivalent process, the rates for input and output of particles are equal to $\tilde{\alpha} = \beta/q$, implying—β being small—that the system is at coexistence between a low density phase with a density of particles equal to $\tilde{\alpha}$ and a high density phase with a density of particles equal to $1 - \tilde{\alpha}$, as reviewed in the introduction. The instantaneous configuration of the system consists of two regions of order N, one of low density of (+) (or high density of (−)) on the left, separated from another one of high density of (+) (or low density of (−)) on the right by a *shock*. The location of the shock diffuses between 0 and N. In the toy model, the location of the random walker is precisely that of the shock. It is uniformly distributed over the whole system. As said in the introduction, this case does not correspond to spontaneous symmetry breaking.

In the other extreme case $b = 0$, i.e., $\alpha = 0$, the dynamics *stops* in one of the states of broken symmetry. Hence this situation does not correspond to genuine spontaneous symmetry breaking.

6. The general case

We now turn to the analysis of the large-N behaviour of the stationary probabilities $p_{j,k}$ and of the average time $\langle T_0 \rangle$ between two flips in the general situation ($0 < a < \frac{1}{2}, b = \frac{1}{2} - a$).

6.1. The stationary probability measure

The stationary probability measure $p_{j,k}$ is the solution of equation (4.2), with the boundary conditions (4.3). It is convenient to use as coordinates j and the number of holes v, such that $j + k + v = N$. With the notation $p_j^{(v)} = p_{j,k}$, equation (4.2) reads

$$p_j^{(v)} = a(p_j^{(v-1)} + p_{j+1}^{(v-1)}) + b(p_{j+1}^{(v)} + p_{j-1}^{(v)}). \tag{6.1}$$

with boundary conditions inherited from equation (4.3). It is shown in appendix B how this equation can be solved numerically in a recursive way, according to increasing values of v, starting from $v = 0$.

In the asymptotic $N \to \infty$ regime of interest, the probability measure is expected to be concentrated in two finite regions around the points $(0, N)$ and $(N, 0)$, each region being decoupled from the other one. This phenomenon, already observed in section 5 for $b = 0$, is a consequence of the bias present in the dynamical rules of the toy model.

Because of the symmetry property (4.4), it is sufficient to consider the vicinity of the point $(0, N)$. We are thus led to look for an asymptotic solution $p_j^{(v)}$ of equation (6.1), independent of N in the $N \to \infty$ limit, and localized, namely decreasing as either j or v gets large, i.e. for $1 \ll j, v \ll N$. The normalization of the probability measure now reads

$$\sum_{v \geqslant 0} \sum_{j \geqslant 0} p_j^{(v)} = \tfrac{1}{2} \tag{6.2}$$

since the two decoupled regions around $(0, N)$ and around $(N, 0)$ bear equal weights.

In order to study these asymptotic probabilities $p_j^{(v)}$, it is convenient to introduce the one-variable generating functions

$$P_j(x) = \sum_{v \geqslant 0} p_j^{(v)} x^v. \tag{6.3}$$

Equation (6.1) then leads to the recursion relation

$$P_j(x) = ax[P_j(x) + P_{j+1}(x)] + b[P_{j+1}(x) + P_{j-1}(x)] \tag{6.4}$$

with (see equation (4.3))

$$P_0(x) = p_0^{(0)}. \tag{6.5}$$

The general solution of equation (6.4) reads

$$P_j(x) = A(x)[z_-(x)]^j + B(x)[z_+(x)]^j \tag{6.6}$$

where $z_\pm(x)$ are the two solutions of the characteristic equation

$$(ax + b)z^2 - (1 - ax)z + b = 0 \tag{6.7}$$

hence

$$z_\pm(x) = \frac{1 - ax \pm \sqrt{\Delta(x)}}{2(ax + b)} \tag{6.8}$$

where

$$\Delta(x) = a^2 x^2 - 4a(1-a)x + 4a(1-a). \tag{6.9}$$

This expression vanishes for $x = x_1$ and $x = x_2$, with

$$x_1 = \frac{2}{a}\left(1 - a - \sqrt{(1-a)(1-2a)}\right) \qquad x_2 = \frac{2}{a}\left(1 - a + \sqrt{(1-a)(1-2a)}\right). \tag{6.10}$$

The associated values of z read

$$z_1 = z_\pm(x_1) = \frac{1 - 2a + 2\sqrt{(1-a)(1-2a)}}{3 - 2a}$$

$$z_2 = z_\pm(x_2) = \frac{1 - 2a - 2\sqrt{(1-a)(1-2a)}}{3 - 2a}. \tag{6.11}$$

The branch $z_-(x)$ is monotonically increasing from $z_- = \exp(-\sigma)$ for $x = 0$, with the notation

$$\cosh\sigma = \frac{1}{2b} = \frac{1}{1-2a} \qquad \text{i.e. } \exp(\pm\sigma) = \frac{1 \pm \sqrt{4a(1-a)}}{1-2a} \tag{6.12}$$

to $z_- = z_1$ for $x = x_1$. Conversely, $z_+(x)$ is monotonically decreasing from $z_+ = \exp(\sigma)$ for $x = 0$ to $z_+ = z_1$ for $x = x_1$.

Since the generating functions $P_j(x)$ are analytic at least in a neighbourhood of the origin $x = 0$, and decrease to zero for large j, the second term in the general solution (6.6) is ruled out. Taking into account the boundary condition (6.5) at $j = 0$, we are left with

$$P_j(x) = p_0^{(0)}[z_-(x)]^j. \tag{6.13}$$

Then, setting $x = 1$ in equation (6.13), and using $z_-(1) = 1 - 2a$, yields

$$P_j(1) = \sum_{\nu \geqslant 0} p_j^{(\nu)} = p_0^{(0)}(1-2a)^j. \tag{6.14}$$

By summing this expression over j, and using equation (6.2), one gets the very simple result

$$p_0^{(0)} = a. \tag{6.15}$$

The average time between two bounces $\langle T_b \rangle$ in the $N \to \infty$ regime can then be evaluated by means of equation (4.16). We thus obtain

$$\langle T_b \rangle = \frac{1}{2ab}. \tag{6.16}$$

Equation (6.13) is the main result of this section, from which the following consequences can be derived.

(i) A quantitative characterization of the concentration of the probability measure around the point $(0, N)$ (and similarly around $(N, 0)$) is provided by the following order parameter

$$M_2(N) = \left\langle \left(\frac{j-k}{N} \right)^2 \right\rangle \tag{6.17}$$

defined as an average with respect to the stationary measure. The existence of the asymptotic probabilities $p_j^{(\nu)}$, independent of N, implies

$$M_2(N) = 1 - \frac{\lambda}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with } \lambda = 4\langle j \rangle + 2\langle \nu \rangle \tag{6.18}$$

the averages being taken with respect to the $p_j^{(\nu)}$. The result (6.13) yields

$$\langle j \rangle = 2 \sum_{j \geqslant 0} j\, P_j(1) = \frac{1-2a}{2a} \tag{6.19}$$

and

$$\langle \nu \rangle = 2 \sum_{j \geqslant 0} \left(\frac{\mathrm{d}P_j(x)}{\mathrm{d}x}\right)_{x=1} = \frac{(1-a)(1-2a)}{a} \tag{6.20}$$

so that we have

$$\lambda = \frac{2(1-2a)(2-a)}{a}. \tag{6.21}$$

Note the simple interpretation of (6.19): $\langle j \rangle$, the average number of $(-)$, i.e. of minority particles in the steady state, is equal to the input rate α. It is also equal to D_{xy}/V_x, using equations (3.5), (3.6).

(ii) From the expression (6.13) of $P_j(x)$ it is also possible to extract the $p_j^{(\nu)}$ by means of the following contour integral encircling the origin in the complex x-plane:

$$p_j^{(\nu)} = \oint \frac{\mathrm{d}x}{2\mathrm{i}\pi} \frac{P_j(x)}{x^{\nu+1}}. \tag{6.22}$$

We will restrict ourselves to the asymptotic behaviour of these probabilities, when both j and ν are large ($1 \ll j, \nu \ll N$), at a fixed angle $0 \leqslant \varphi \leqslant \pi/4$ between the negative k-axis and the vector joining $(0, N)$ to (j, k), namely we set

$$\begin{cases} N - k = j + \nu = \rho \cos \varphi \\ j = \rho \sin \varphi. \end{cases} \tag{6.23}$$

We are thus led to evaluate the integral

$$p_j^{(\nu)} = a \oint \frac{\mathrm{d}x}{2\mathrm{i}\pi x} \exp\{-\rho[(\cos \varphi - \sin \varphi) \ln x - \sin \varphi \ln z_-(x)]\} \tag{6.24}$$

for φ fixed, in the $\rho \to \infty$ limit. The direction $\varphi = 0$ corresponds to $\nu \to \infty$ at constant j, whereas $\varphi = \pi/4$ corresponds to $j \to \infty$ at constant ν.

A saddle-point calculation leads to the estimate

$$p_j^{(\nu)} \sim \exp\{-\rho\mu(\varphi)\} \tag{6.25}$$

with

$$\mu(\varphi) = (\cos \varphi - \sin \varphi) \ln x_{\mathrm{c}} - \sin \varphi \ln z_-(x_{\mathrm{c}}) \tag{6.26}$$

and the saddle point $x_c(\varphi)$ is given by

$$\cot\varphi = 1 + \frac{x_c}{z_-(x_c)} \left(\frac{dz_-(x)}{dx}\right)_{x=x_c}.$$ (6.27)

We have thus demonstrated that the asymptotic probabilities $p_j^{(\nu)}$ decay exponentially in all directions away from the point $(0, N)$, with an inverse decay length given by the *mass* $\mu(\varphi)$ in the direction defined by the angle φ. Equations (6.26) and (6.27) provide a parametric representation of $\mu(\varphi)$. It increases monotonically between $\varphi = 0$, where $x_c = x_1$, and

$$\mu(0) = \mu = \ln x_1$$ (6.28)

with the notation (6.10), and $\varphi = \pi/4$, where $\dot{x}_c = 0$, and $\mu(\pi/4) = \sigma/\sqrt{2}$, with the notation (6.12).

6.2. The average time between two flips $\langle T_0 \rangle$

This quantity is given by equation (4.13). First, the asymptotic behaviour of $u_{1,N-1}$ for large N is easy to determine. Indeed the walker will visit the j-axis with a vanishingly small probability. One may therefore neglect the presence of this boundary, and just consider the displacements of the walker parallel to the j-axis. Thus $u_{j,k}$ reduces to u_j, the average time for the walker starting at site j to reach the boundary, i.e. site $j = 0$. During $d\tau$ the walker may either hop to the left with probability $(a+b)d\tau$ or to the right with probability $bd\tau$ or stay on site j with probability $ad\tau$. Therefore the set of u_j obeys the equation

$$u_j = 1 + (a+b)u_{j-1} + bu_{j+1} + au_j$$ (6.29)

with the boundary condition $u_0 = 0$. Equivalently equation (6.29) is obtained by dropping the k-dependency in equation (4.10a). The solution to equation (6.29) is simply $u_j = j/a$. Coming back to the original two-dimensional situation, we find

$$u_{1,N-1} \to \frac{1}{a} \qquad (N \gg 1).$$ (6.30)

This result provides an alternative derivation of equation (6.15), by means of equation (4.17).

The analysis of the asymptotic behaviour of $v_{j,1}$ is more difficult. Introducing the generating function $V(x, y)$ as in equation (5.5), we see that equation (4.10b) is equivalent to

$$D(x, y)V(x, y) + bxy[xV_1(x) + yV_2(y)] = \frac{x^2y^2}{2(1-y)}$$ (6.31)

where

$$\begin{cases} V_1(x) = \frac{\partial V}{\partial y}(x, y)_{y=0} = \sum_{j=1}^{\infty} v_{j,1}x^j \\ V_2(y) = \frac{\partial V}{\partial x}(x, y)_{x=0} = \sum_{k=1}^{\infty} v_{1,k}y^k \end{cases}$$ (6.32)

and

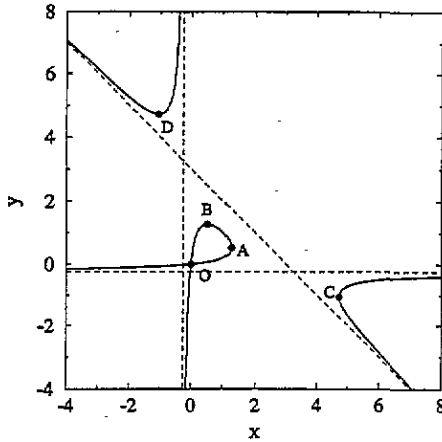$$D(x, y) = xy - a(x+y)xy - b(x^2+y^2).$$ (6.33)

Figure 1. Characteristic curve $D(x, y) = 0$ for $a = 0.4$ (see equation (6.33)). The broken curves are the asymptotes.

The characteristic equation $D(x, y) = 0$ defines the characteristic curve of the problem. It is a rational cubic curve, with the origin as a double point, and the three lines $x = -b/a$, $y = -b/a$, and $x + y = (1 + 2b)/a$ as asymptotes. Figure 1 depicts the characteristic curve for $a = 0.4$, $b = 0.5 - a = 0.1$.

We notice that the characteristic equation is equivalent to equation (6.7), up to the substitution $y = xz$. Solving the characteristic equation for y, we thus get

$$y_\pm(x) = xz_\pm(x) \tag{6.34}$$

with the notation (6.8).

The characteristic curve has either a horizontal or a vertical tangent at the four points marked on figure 1. Their coordinates read $A = (x_1, y_1)$, $B = (y_1, x_1)$, $C = (x_2, y_2)$, $D = (y_2, x_2)$, where $y_1 = x_1 z_1$, $y_2 = x_2 z_2$, with the notation (6.10) and (6.11). Figure 1 also provides a geometric illustration of the saddle-point approach to the stationary probability, presented in section 6.1. Indeed, as the angle φ varies from 0 to $\pi/4$, the saddle point runs along the dispersion curve, in coordinates $(x_c, y_c) = (x_c, x_c z_-(x_c))$, from the point A to the origin O.

Let us come back to the $v_{j,k}$. Since these numbers are bounded, the function $V(x, y)$ is analytic in a complex domain $\mathcal{D}$ containing at least the product of disks $|x| < 1$, $|y| < 1$. Whenever x and y are in $\mathcal{D}$, and related by the characteristic equation $D(x, y) = 0$, equation (6.31) simplifies to

$$x V_1(x) + y V_2(y) = \frac{xy}{2b(1 - y)}. \tag{6.35}$$

This equation shows that at least one of the two functions $V_1(x)$ or $V_2(y)$ becomes singular when the relation between x and y becomes itself singular, i.e. at the marked points mentioned above, as may be seen by taking the derivatives of both sides.

The points A and B are the relevant ones, since they are the closest to the origin. Since these two points play symmetric roles, we can consider point A for definiteness. We have $y_1 < 1 < x_1$, so that $V_1(x)$ is singular at $x = x_1$, whereas $V_2(y)$ is regular at $y = y_1$.

We set

$$x = x_1 - \epsilon_x \qquad y = y_1 + \epsilon_y \tag{6.36}$$

so that the characteristic equation reads, to leading order in ϵ_x and ϵ_y,

$$\epsilon_x \approx \left(\frac{ax_1+b}{x_1}\right)^2 \frac{\epsilon_y^2}{aw} \qquad \text{with } w = \sqrt{(1-a)(1-2a)}. \tag{6.37}$$

By inserting this estimate into equation (6.35), and expanding both sides to leading order in ϵ_y, we are led to the conclusion that $V_1(x)$ has a singular part of the form

$$V_1(x) \approx V_1(x_1) - K\sqrt{x_1 - x} \tag{6.38}$$

with

$$K = \frac{\sqrt{aw}}{ax_1+b} \frac{\partial}{\partial y}\left[yV_2(y) - \frac{x_1 y}{2b(1-y)} \right]_{y=y_1}. \tag{6.39}$$

By inserting the singular behaviour (6.38) into the definition (6.32) of $V_1(x)$, we get the estimate

$$v_{j,1} \approx \frac{K\exp(\mu/2)}{2\sqrt{\pi}} j^{-3/2} \exp(-j\mu) \qquad (j \gg 1) \tag{6.40}$$

where μ has been defined in equation (6.28).

We can now insert both estimates (6.30) and (6.40) into the expression (4.13) of $\langle T_0 \rangle$. We thus obtain

$$\langle T_0 \rangle \approx L\, N^{3/2} \exp(N\mu) \qquad (N \gg 1) \tag{6.41}$$

with

$$L = \frac{2\sqrt{\pi}}{a(1-2a)K\exp(3\mu/2)}. \tag{6.42}$$

We have thus derived the full asymptotic expression of the average time between two flips.

• The mass μ is given by equations (6.10), (6.28). It vanishes for small values of a as $\mu = a/4 + 11a^2/32 + \cdots$. This regime corresponds to a continuum limit, to be investigated in detail in section 7. The opposite limit $a \to \frac{1}{2}$ yields $\mu \to \ln 2$, in agreement with equation (5.12). The mass increases monotonically between these two limits. For instance in the case $a = b = \frac{1}{4}$ one finds $\mu = \ln(2(3-\sqrt{6})) = 0.096\,237$.

• The exponent $\frac{3}{2}$ is universal, i.e. independent of a, except for the two limiting cases $a = 0$ and $b = 0$, investigated in section 5. The general analysis presented above fails in these cases, because the characteristic curve is degenerate. We have indeed $D(x, y) = -(x-y)^2/2$ for $a = 0$, and $D(x, y) = xy(2-x-y)/2$ for $b = 0$.

• The absolute prefactor L only depends on a. It cannot be determined within the present approach. The analysis of the continuum limit will provide a quantitative estimate of the behaviour of this prefactor as $a \to 0$. Indeed equation (7.45) yields $L \approx 2/\sqrt{\pi^3 a}$, i.e. $K \approx \pi^2/\sqrt{a}$.

7. The continuum limit

In this section we determine the scaling behaviour of $u_{j,k}$, $v_{j,k}$ and $\langle T_0 \rangle$ in the continuum limit, corresponding to the regime of a small bias ($a \rightarrow 0$). It will turn out that this analysis provides a complete description of the crossover between the diffusive power law (5.3) and the exponential law (6.41).

We consider first the probability of ruin $v_{j,k}$. We assume that $v_{j,k}$ is a smooth function of the continuous rescaled variables $X = ja$ and $Y = ka$. Expanding equation (4.10b) to the lowest non-trivial order in a leads to

$$\left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}\right)^2 v(X, Y) = 2\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right) v(X, Y) \tag{7.1}$$

hence to

$$\frac{\partial^2 v(\xi, \theta)}{\partial \xi^2} = \frac{\partial v(\xi, \theta)}{\partial \theta} \tag{7.2}$$

with

$$\begin{cases} \xi = X - Y = (j - k)a \\ \theta = X + Y = (j + k)a. \end{cases} \tag{7.3}$$

We thus obtain a heat equation, or diffusion equation, where ξ and θ are respectively the space-like and time-like variables. Note that time-like axis is oriented along the direction of the bias. This partial differential equation inherits its boundary conditions from equations (4.11a, b):

$$v(\xi = -\theta, \theta) = 1 \qquad v(\xi = \theta, \theta) = 0. \tag{7.4}$$

These moving boundary conditions are difficult to handle. It is convenient to introduce the dimensionless space-like variable

$$x = \frac{k}{j + k} \tag{7.5}$$

such that

$$\xi = (1 - 2x)\theta. \tag{7.6}$$

Equation (7.2) thus transforms into

$$\frac{\partial^2 v}{\partial x^2} - 2(1 - 2x)\theta\frac{\partial v}{\partial x} - 4\theta^2\frac{\partial v}{\partial \theta} = 0 \tag{7.7}$$

with the boundary conditions

$$v(x = 0, \theta) = 0 \qquad v(x = 1, \theta) = 1. \tag{7.8}$$

The Gaussian

$$G(x, \theta) = (4\pi\theta)^{-1/2}\exp(-\theta(x - \tfrac{1}{2})^2) \tag{7.9}$$

is an elementary solution of equation (7.7), neglecting boundary conditions. We are therefore led to the change of function

$$v(x, \theta) = G(x, \theta) F_v(x, \theta) \tag{7.10}$$

which inserted into equation (7.7) gives

$$\frac{\partial^2 F_v}{\partial x^2} - 4\theta^2 \frac{\partial F_v}{\partial \theta} = 0. \tag{7.11}$$

A last change on the time-like variable

$$\theta = -\frac{1}{4\varepsilon} \tag{7.12}$$

leads to a heat equation for F_v,

$$\frac{\partial^2 F_v}{\partial x^2} = \frac{\partial F_v}{\partial \varepsilon} \tag{7.13}$$

with boundary conditions

$$F_v(x = 0, \varepsilon) = 0 \qquad F_v(x = 1, \varepsilon) = f_v(\varepsilon) = \left(\frac{\pi}{-\varepsilon}\right)^{1/2} \exp\left(-\frac{1}{16\varepsilon}\right). \tag{7.14}$$

and with $\lim_{\varepsilon \to -\infty} F_v(x, \varepsilon) = 0$.

We thus have to solve the heat equation on the unit interval, with prescribed time-dependent boundary conditions. The solution is given by the convolution

$$F_v(x, \varepsilon) = \int_0^\infty d\varepsilon' \, f_v(\varepsilon - \varepsilon') g_v(x, \varepsilon') \tag{7.15}$$

where

$$g_v(x, \varepsilon) = 2\pi \sum_{k=1}^\infty (-1)^{k-1} k \sin(k\pi x) \exp(-k^2\pi^2\varepsilon) \tag{7.16}$$

is the appropriate Green function of the heat equation (7.13). We mention for further reference its alternative expression, obtained from equation (7.16) by the Poisson formula

$$g_v(x, \varepsilon) = (4\pi\varepsilon^3)^{-1/2} \sum_{m=-\infty}^\infty (2m + 1 - x) \exp\left(-\frac{(2m + 1 - x)^2}{4\varepsilon}\right). \tag{7.17}$$

We are thus left with the following explicit formula for the scaling behaviour of the $v_{j,k}$ in the continuum limit

$$v_{j,k} \approx v(x, \theta) = \exp(-\theta(x - \tfrac{1}{2})^2) \int_0^\infty \frac{d\varepsilon}{(1 + 4\theta\varepsilon)^{1/2}} \exp\left(\frac{\theta}{4(1 + 4\theta\varepsilon)}\right) g_v(x, \varepsilon). \tag{7.18}$$

We recall that the scaling variables x and θ are related to j and k by equations (7.3), (7.5).

We are especially interested in the quantity $v_{N-1,1}$, which enters the expression (4.13) of $\langle T_0 \rangle$. Equation (7.18) yields

$$v_{N-1,1} \approx a\Phi(\theta) \qquad (\theta = Na) \tag{7.19}$$

where $\Phi(\theta)$ is the following scaling function

$$\Phi(\theta) = \frac{1}{\theta}\frac{\partial}{\partial x}v(x=0,\theta) = \frac{1}{\theta}\int_0^\infty \frac{d\varepsilon}{(1+4\theta\varepsilon)^{1/2}}\exp\left(-\frac{\theta^2\varepsilon}{1+4\theta\varepsilon}\right)g_0(\varepsilon) \tag{7.20}$$

with

$$g_0(\varepsilon) = \frac{\partial}{\partial x}g_v(x=0,\varepsilon)$$

$$= 2\pi^2\sum_{k=1}^\infty(-1)^{k-1}k^2\exp(-k^2\pi^2\varepsilon) \tag{7.21}$$

$$= (4\pi\varepsilon^3)^{-1/2}\sum_{m=-\infty}^\infty\left(\frac{(2m+1)^2}{2\varepsilon}-1\right)\exp\left(-\frac{(2m+1)^2}{4\varepsilon}\right).$$

The scaling behaviour of the duration of the game $u_{j,k}$ can be investigated in a similar way. Expanding equation (4.10a) now leads to an inhomogeneous heat equation,

$$\frac{\partial u(\xi,\theta)}{\partial\theta} - \frac{\partial^2 u(\xi,\theta)}{\partial\xi^2} = \frac{1}{2a^2} \tag{7.22}$$

with the moving boundary conditions

$$u(\xi=-\theta,\theta) = u(\xi=\theta,\theta) = 0. \tag{7.23}$$

We again use the dimensionless space-like variable x, setting

$$u(x,\theta) = G(x,\theta)F_u(x,\theta) \tag{7.24}$$

and we change the time-like variable from θ to ε. We thus obtain an inhomogeneous heat equation for F_u,

$$\frac{\partial F_u}{\partial\varepsilon} - \frac{\partial^2 F_u}{\partial x^2} = f_u(x,\varepsilon) \tag{7.25}$$

with boundary conditions

$$F_u(x=0,\varepsilon) = F_u(x=1,\varepsilon) = 0 \tag{7.26}$$

and with $\lim_{\varepsilon\to-\infty}F_u(x,\varepsilon) = 0$. The source term reads

$$f_u(x,\varepsilon) = \frac{1}{8a^2}\left(\frac{\pi}{(-\varepsilon)^5}\right)^{1/2}\exp\left(-\frac{(2x-1)^2}{16\varepsilon}\right). \tag{7.27}$$

We now have to solve the heat equation on an interval, with a prescribed time-dependent source, and Dirichlet boundary conditions. The solution is again given by a convolution,

$$F_u(x, \varepsilon) = \int_0^\infty d\varepsilon' \int_0^1 dy \, f_u(y, \varepsilon - \varepsilon') \, g_u(x, y, \varepsilon') \tag{7.28}$$

where

$$g_u(x, y, \varepsilon) = 2 \sum_{k=1}^\infty \sin(k\pi x) \sin(k\pi y) \exp(-k^2\pi^2\varepsilon) \tag{7.29}$$

is the appropriate Green function of the heat equation.

We are thus left with the following explicit formula for the scaling behaviour of the $u_{j,k}$ in the continuum limit:

$$u_{j,k} \approx u(x, \theta) = \frac{2\theta^2}{a^2} \exp(-\theta(x - \tfrac{1}{2})^2)$$

$$\times \int_0^1 dy \int_0^\infty \frac{d\varepsilon}{(1 + 4\theta\varepsilon)^{5/2}} \exp\left(\frac{\theta(2y - 1)^2}{4(1 + 4\theta\varepsilon)}\right) g_u(x, y, \varepsilon). \tag{7.30}$$

We are especially interested in the quantity $u_{N-1,1}$, which enters expression (4.13) of $\langle T_0 \rangle$. Equation (7.30) yields

$$u_{N-1,1} \approx \frac{1}{a}\Psi(\theta) \qquad (\theta = Na) \tag{7.31}$$

where $\Psi(\theta)$ is the following scaling function

$$\Psi(\theta) = \frac{a^2}{\theta} \frac{\partial}{\partial x} u(x = 0, \theta)$$

$$= 2\theta \int_0^1 dy \int_0^\infty \frac{d\varepsilon}{(1 + 4\theta\varepsilon)^{5/2}} \exp\left(\frac{(y^2 - y - \theta\varepsilon)\theta}{1 + 4\theta\varepsilon}\right) g_v(1 - y, \varepsilon) \tag{7.32}$$

where we have used the identity

$$\frac{\partial}{\partial x} g_u(x = 0, y, \varepsilon) = g_v(1 - y, \varepsilon). \tag{7.33}$$

Putting together both results (7.19), (7.31) we obtain the following expression for the scaling behaviour of the average time between two flips

$$\langle T_0 \rangle \approx \frac{1}{a^2} \Delta(\theta) \qquad (\theta = Na) \tag{7.34}$$

where the scaling function $\Delta(\theta)$ reads

$$\Delta(\theta) = \frac{\Psi(\theta)}{\Phi(\theta)}. \tag{7.35}$$

The regimes of small and large values of the time-like scaling variable $\theta = Na$ are of special interest. We now show how the scaling law (7.34) describes the crossover between the diffusive law (5.3) in the unbiased case ($\theta \ll 1$), and the exponential law (6.41) in the generic biased case ($\theta \gg 1$).

7.1. $\theta \ll 1$

In this regime it is most convenient to consider equations (7.2), (7.22) and to look for solutions to them in the form of power series in θ. We thus obtain

$$
\begin{cases}
u(x,\theta) = \dfrac{\theta^2}{a^2} x(1-x)\left(1 - \theta + \dfrac{\theta^2}{3}(4 + x - x^2) + \cdots\right) \\
v(x,\theta) = x + x(1-x)(1-2x)\left(-\dfrac{\theta}{3} + \dfrac{\theta^2}{15}(1 - 2x + 2x^2) + \cdots\right).
\end{cases}
\tag{7.36}
$$

The leading terms of the above expressions agree with the results (5.2) corresponding to $a = 0$. The above results also yield the following expansions for the scaling functions

$$
\begin{cases}
\Phi(\theta) = \dfrac{1}{\theta}\left(1 - \dfrac{\theta}{3} + \dfrac{\theta^2}{15} + \cdots\right) \\
\Psi(\theta) = \theta\left(1 - \theta + \dfrac{4\theta^2}{3} + \cdots\right) \\
\Delta(\theta) = \theta^2\left(1 - \dfrac{2\theta}{3} + \dfrac{47\theta^2}{45} + \cdots\right).
\end{cases}
\tag{7.37}
$$

The last of these expansions yields more explicitly

$$
\langle T_0\rangle \approx N^2\left(1 - \frac{2Na}{3} + \frac{47N^2a^2}{45} + \cdots\right) \qquad (N \gg 1, Na \ll 1).
\tag{7.38}
$$

The leading term reproduces the asymptotic behaviour of the exact result (5.3), characteristic of diffusive motion. Note that the first correction term to this law is negative. This may be understood as follows. In the current regime (i.e. $\alpha \gg N \gg 1$), we typically expect a small number of zeros to separate the $(+)$ and $(-)$ blocks, thus leading to a small negative correction to the average time between two flips of the strictly diffusive shock behaviour.

7.2. $\theta \gg 1$

This regime is the most relevant physically, since it corresponds to the asymptotic behaviour (6.41). The asymptotic behaviour of the scaling functions can be determined from the integral expressions (7.20), (7.32).

The calculation for $\Psi(\theta)$ is as follows. The double integral in equation (7.32) is dominated by the region where both variables ϵ and y are small. In this regime the function $g_v(1 - y, \varepsilon)$ can be approximated by the $m = 0$ term in the sum (7.17). Choosing the rescaled integration variables $z = y/(\theta\varepsilon)$ and $\zeta = \theta^2\varepsilon$, we are left with

$$
\Psi(\theta) \approx \frac{1}{\sqrt{\pi}} \int_0^\infty z\,\mathrm{d}z \int_0^\infty \frac{\zeta^{1/2}\,\mathrm{d}\zeta}{(1 + 4\zeta/\theta)^{5/2}} \exp\left(-\frac{(z+2)^2\zeta}{4(1+4\zeta/\theta)}\right).
\tag{7.39}
$$

The ζ-integral can be performed by changing variable from ζ to $\eta = \zeta/(1 + 4\zeta/\theta)$. After some algebra we are left with the very simple limit behaviour

$$
\Psi(\theta) \to 1 \qquad (\theta \gg 1)
\tag{7.40}
$$

in agreement with the more general result (6.30). The correction to this limit vanishes exponentially as $\exp(-\theta/4)$, since it is due to the upper bound $\eta_{max} = \theta/4$ of the η-integral.

Let us now compute $\Phi(\theta)$. We perform the change of variable $\zeta = \theta/(1 + 4\theta\varepsilon)$ in the integral representation (7.20), with the last expression of equation (7.21) for the function g_0, obtaining thus

$$\Phi(\theta) \approx \frac{2\exp(-\theta/4)}{(\pi\theta^3)^{1/2}} I \tag{7.41}$$

with

$$I = \sum_{m=0}^{\infty} \int_0^{\infty} d\zeta \, (2(2m+1)^2 \zeta - 1) \exp[-((2m+1)^2 - \tfrac{1}{4})\zeta]$$

$$= \sum_{m=0}^{\infty} \frac{(2m+1)^2 + \tfrac{1}{4}}{((2m+1)^2 - \tfrac{1}{4})^2} = 2 \sum_{m=0}^{\infty} \left(\frac{1}{(4m+1)^2} + \frac{1}{(4m+3)^2} \right) = \frac{\pi^2}{4}. \tag{7.42}$$

We thus obtain the estimate

$$\Phi(\theta) \approx \frac{1}{2} \left(\frac{\pi}{\theta} \right)^{3/2} \exp(-\theta/4) \qquad (\theta \gg 1). \tag{7.43}$$

The asymptotic estimates (7.40) and (7.43) yield

$$\Delta(\theta) \approx 2 \left(\frac{\theta}{\pi} \right)^{3/2} \exp(\theta/4) \qquad (\theta \gg 1) \tag{7.44}$$

namely

$$\langle T_0 \rangle \approx 2 \left(\frac{N^3}{a\pi^3} \right)^{1/2} \exp(Na/4) \qquad (Na \gg 1). \tag{7.45}$$

This result exhibits an exponential growth with mass $\mu = a/4$ and a power-law prefactor in $N^{3/2}$. These results are in agreement with the analysis of section 6.

In conclusion we see that the analysis contained in this section fully describes the crossover from the diffusive behaviour of a shock ($\theta \ll 1$, i.e. $\alpha \gg N \gg 1$) to the symmetry breaking regime ($\theta \gg 1$, i.e. $N \gg \alpha \gg 1$). This crossover behaviour is illustrated on figure 2, showing a logarithmic plot of $\langle T_0 \rangle$ against $\theta = Na$. The numerical data for $a = 0.1$, 0.06, and 0.03, obtained by means of the approach described in appendix B, smoothly converge toward the result (7.35) of the continuum limit. The smoothness of the corrections to the continuum limit has already been noticed below equation (6.42), for the mass μ.

It is striking to observe the richness of the behaviour of the original exclusion process: even in the regime studied here, i.e. $\beta \to 0$ and $\alpha \to \infty$, the curve obtained for $\langle T_0 \rangle$ is very similar to that obtained in [4] by numerical simulations of the exclusion process with finite values of α and β (see figure 4 of that work).

Finally we notice that the stationary probability measure $p_{j,k}$ also exhibits a scaling behaviour in the continuum limit. A detailed investigation of this behaviour is not needed. We just mention that equation (4.17) implies the following scaling law for the probability at both endpoints $(0, N)$ and $(N, 0)$, in the regime $a \ll 1$, $N \gg 1$,

$$p_{0,N} = p_0^{(0)} \approx \frac{a}{\Psi(\theta)} \qquad (\theta = Na). \tag{7.46}$$

As expected, this result interpolates between the $a = 0$ limit, equation (5.1), and the $N \to \infty$ regime, equation (6.15).
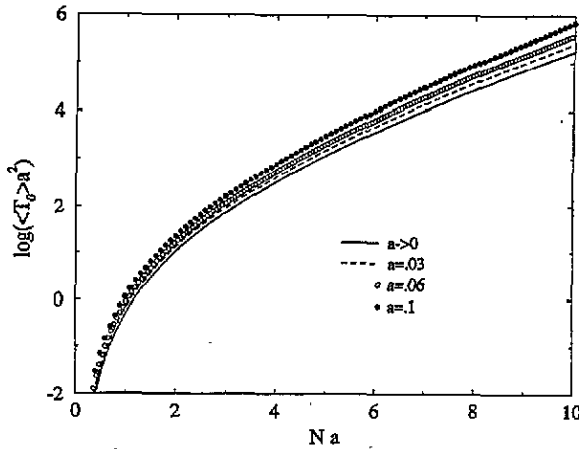
Figure 2. Logarithmic plot of $\langle T_0 \rangle a^2$ as a function of Na for the continuum limit (full curve) and for the discrete random walk, with $a = 0.1, 0.06, 0.03$.

8. Conclusion

The analysis presented in this paper confirms the intuitive picture, given at the end of section 3, of the mechanism leading to spontaneous symmetry breaking in the toy model, and quantifies it in several respects.

First, the stationary probability measure $p_{j,k}$ is localized in a symmetric way around the endpoints $(0, N)$ and $(N, 0)$, in the asymptotic regime of a large system size N. This is intuitively clear: the bias of the random walker is in the south-west direction so that its drift is towards whichever boundary is nearest. It then becomes localized near one of the endpoints due to the bouncing property of the boundary. This localization is quantified by the behaviour (6.18) of the order parameter $M_2(N)$, and by the exponential decay (6.25) of the stationary probabilities in all directions away from both endpoints. The mass, or inverse decay length, $\mu(\varphi)$ depends on the orientation: it decreases monotonically between $\varphi = \pi/4$ (decay along the diagonal $j + k = N$) and $\varphi = 0$ (decay along both coordinate axes).

This point brings some more insight on the flipping mechanism. Since the decay length is the largest along the axes, the walker, starting from $(0, N)$, will preferentially diffuse down the k-axis, until it possibly reaches the vicinity of the origin, corresponding in the toy model to few $(-)$, few $(+)$ and many holes in the system. Then, in a few steps, it can pass the line $j = k$ and reach a region where the bias drives it to the j-axis. The typical duration of such a successful path, from $(0, N)$ down to the j-axis, is presumably of order N, indicating that, when the system flips, it does so on a very short time scale compared to the average duration of time it spends in a long lived state, i.e. near the two endpoints $(0, N)$ or $(N, 0)$.

This argument also provides an estimate of the average time between two flips $\langle T_0 \rangle$, i.e. the average lifetime of the walker in one of the two regions near the endpoints. Indeed the probability that the walker reaches the vicinity of the origin is proportional to $\exp(-N\mu)$, with $\mu = \mu(0)$, therefore $\langle T_0 \rangle$ should be proportional to $\exp(N\mu)$.

We indeed find that the average time $\langle T_0 \rangle$ between two flips in the toy model reads

$$\langle T_0 \rangle \approx LN^{3/2} \exp(N\mu) \tag{8.1}$$

with

$$\mu = \mu(0) = \ln x_1 = \ln \left[\frac{2}{a} \left(1 - a - \sqrt{(1-a)(1-2a)} \right) \right].$$ (8.2)

This result agrees with the estimate (1.4). It also confirms the above picture of the flipping mechanism. The mass μ can be interpreted as an activation energy, or barrier height per particle; the exponent $\frac{3}{2}$ of the power-law prefactor is universal, i.e., it does not depend on the strength of the bias; the absolute prefactor L is only known in the continuum limit of a weak bias ($a \to 0$), where $L \approx 2/\sqrt{a\pi^3}$.

We have thus demonstrated the occurrence of spontaneous symmetry breaking in the $\beta \to 0$ limit of the two-species exclusion model, in the sense that the system flips between two symmetry-related long lived states, by showing that the two properties mentioned above hold.

Let us now impose boundary conditions which favour one species of particles over the other, breaking thus the symmetry explicitly. We consider an exclusion model where, instead of having boundary rates α, β for both types of particles, we have α, β_+ for the positive particles and α, β_- for the negative particles. We set $\beta_\pm = \beta(1 \mp H)$, where H is a symmetry-breaking field, such that $|H| < 1$. Hence the positive (respectively, negative) particles are favoured by the boundary conditions for $H > 0$ (respectively, $H < 0$). If we now take $\beta \to 0$, it can easily be checked that we find a toy model with the following hopping rates: $a(1 + H)$ west; $a(1 - H)$ south; $b(1 + H)$ north-west; $b(1 - H)$ south-east, again with $a + b = \frac{1}{2}$.

The bias associated with this model now reads

$$V = a \begin{pmatrix} -1 \\ -1 \end{pmatrix} + (1-a)H \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ (8.3)

As long as the field is small enough, i.e. $|H| < H_0 = a/(1 - a)$, both components of the bias are still negative, and the mechanism which localizes the walker near the endpoints is retained. The significance of this is as follows. Assume now $H > 0$ for definiteness. Even though the boundary conditions favour blocks of positive particles, the system may still spend lengths of time which diverge exponentially with the system size in a state dominated by negative particles. It can indeed be shown that the average durations of time spent in the $(+)$ phase and in the $(-)$ phase scale respectively as $\langle T_0 \rangle_+ \sim \exp(N\mu_+)$ and $\langle T_0 \rangle_- \sim \exp(N\mu_-)$, where $\mu_+ > \mu_-$ are the masses along both axes, which are now different. This reinforces our proposed interpretation of the mass as the activation energy per particle, in the sense that $\Delta\mu = \mu_+ - \mu_-$ is naturally interpreted as the difference in energy per particle between both pure phases. In the continuum limit of the model, defined as $a \to 0$ and $H \to 0$ simultaneously, we have $\mu_\pm \approx (a \pm H)^2/(4a)$, hence $\Delta\mu \approx H$.

The analogy between H and a symmetry-breaking field in equilibrium statistical physics, such as a magnetic field in the Ising model, can be pursued. Consider the order parameter $M_1(N) = \langle (k - j)/N \rangle$, analogous to a magnetization. This quantity can be estimated by noticing that the statistical weights $p_\pm$ of both phases are proportional to the durations of time $\langle T_0 \rangle_\pm$ spent in them, hence $M_1(N) \approx p_+ - p_- \approx \tanh(NH/2)$ within exponential accuracy, the latter formula being identical to the equation of state of the Ising model near coexistence.

We finally notice that the unfavoured phase is metastable for $H < H_0$ and just unstable for $H > H_0$. In the latter case the system will always mostly consist of $(+)$ particles.

We close up with a short discussion on the behaviour of the mass in the continuum limit of a weak bias. We consider the most general biased random walk in the quarter

plane, where the velocity is still given by equation (3.5), while the diffusion coefficients $D_\parallel$ and $D_\perp$ are now different from each other *a priori*. It can be shown that the on-axis mass μ which enters the law (1.4) vanishes linearly with a, with a non-universal prefactor depending on the diffusion coefficients,

$$\mu \approx \frac{a}{2D_\perp(1 + \sqrt{1 + D_\parallel/D_\perp})}. \tag{8.4}$$

In the case of the toy model, with $D_\parallel \approx 0$ and $D_\perp \approx 1$, we recover $\mu \approx a/4$ in the continuum limit. Another case of interest is the north–south–east–west model, defined by moves to the four cardinal directions with rates b (north and west), $a + b$ (south and east), (with $a + 2b = \frac{1}{2}$), for which interesting properties can be derived by means of several approaches [15, 16]. In that case, we have $D_\parallel \approx D_\perp \approx \frac{1}{4}$, hence $\mu \approx 2(\sqrt{2} - 1)a$ in the continuum limit.

Acknowledgments

Appendix A

In this appendix we show that the toy model is the $\beta \to 0$ limit of the two-species exclusion process, in a sense that will be made precise shortly. The argument implies that this will be true even if the time over which the models are compared, and the system size, become infinite as the limit is taken.

We begin with a summary of the basic strategy. Let us write $\eta(t)$ for the configuration of the exclusion process at time t; $\eta(t)$ is determined by the initial configuration $\eta(0)$ together with some specification of the (random) transitions during the evolution. Similarly, $X(\tau) = (j(\tau), k(\tau))$ will denote the position of the walker in the toy model at time τ, and is determined by $X(0)$ and the random steps taken by the walker. Now to each configuration $X = (j, k)$ of the toy model there corresponds a configuration $\Phi(X)$ of the exclusion process: $\Phi(X) \equiv (- \cdots - 0 \cdots 0 + \cdots +)$, with j $(-)$ particles and k $(+)$ particles. We will show that $\eta(t)$ agrees, with a probability which approaches 1 as $\beta \to 0$, with the time-rescaled image $\Phi(X(2\beta t))$ of the toy process, assuming (a) that the processes coincide initially, i.e. $\eta(0) = \Phi(X(0))$, and (b) that the random transitions in the two models correspond. Achieving the latter is called *coupling* the models.

It is convenient to realize the randomness in the exclusion process as follows. There are $3(N - 1) + 4$ distinct types of elementary transitions which occur in this process: three types of exchange ($+$ with $-$, $+$ with 0, or 0 with $-$) for each nearest neighbour site pair, and entrance or exit of a particle at the left or right boundary. With each type of transition we associate a 'Poisson alarm clock', which is in fact just a random set of positive times, Poisson distributed with density equal to the rate—q, 1, α or β—for the transition. The alarm clock 'rings' at each time t^* in this set, and the transition occurs if it is permitted in the configuration $\eta(t^*)$ at that time. (Here and below we assume, for definiteness, that $\eta(t)$ and $X(\tau)$ are left continuous, so that if a transition from configuration ζ to configuration

ζ' occurs at time t, then $\eta(t) = \zeta$ and $\eta(t+) \equiv \lim_{s \searrow t} \eta(s) = \zeta'$.) We will refer to these random times generically as *transition times* and more specifically as *entrance times*, *left exit times*, etc; as emphasized above they are in fact *potential* transition times.

To couple the two models we use the random times from the exclusion process to determine the evolution of the toy model; this coupling corresponds to the intuitive description in section 2. Specifically, if t^* is a left exit time, then $\tau^* = 2\beta t^*$ is a time at which j can decrease in the toy model. Suppose that $X(\tau^*) = (j, k)$. There are several possibilities for $X(\tau^*+) = (j', k')$:

Case I: $j = 0$. Nothing happens in the toy model: $(j', k') = (j, k)$.

Case II: $j = 1$. In this case, $(j', k') = (0, N)$.

Case III: $j \geqslant 2$. The transition in the toy model is determined by looking ahead of t^* to the first transition time $t^{**} > t^*$ for one of two types of transitions: either for the entrance of a + particle, in which case $(j', k') = (j - 1, k + 1)$, or for a 0− → −0 exchange on bond $\langle 1, 2 \rangle$, in which case $(j', k') = (j - 1, k)$, if $k > 0$, and $(j', k') = (N, 0)$, if $k = 0$.

These possibilities are summarized in table A1. Right exit times correspond similarly to times of possible decrease of k in the toy model. Since the relative probabilities of cases III.1 and III.2 are $\alpha/(1 + \alpha)$ and $1/(1 + \alpha)$, this prescription reproduces the transition rules (2)–(4). Here it is important that transitions in the toy model are governed by the alarm clocks, not by the actual occurrence of transitions in the exclusion process, since the latter depend on the history of that process and hence their true probabilities are not known.

Table A1. Transitions in the toy model for which j may decrease. The initial state is $(j, k) = (j(\tau^*), k(\tau^*))$.

j	Allowed transition at time t^{**} in exclusion process	k	Final state $(j(\tau^*+), k(\tau^*+))$	Case
$j = 0$	—	N	$(0, N)$	I
$j = 1$	—	—	$(0, N)$	II
$j \geqslant 2$	$0 - \cdots \to + - \cdots$	—	$(j - 1, k + 1)$	III.1
	$0 - \cdots \to -0 \cdots$	$k > 0$	$(j - 1, k)$	III.2.a
		$k = 0$	$(N, 0)$	III.2.b

Suppose now that we observe the two systems over a time interval $[0, T]$ of the toy model—that is, for $0 \leqslant \tau \leqslant T$ or $0 \leqslant t \leqslant T/(2\beta)$. Let $B \subset [0, T]$ be the set of (bad) times τ at which $\eta(\tau/(2\beta)) \neq \Phi(X(\tau))$, i.e. at which the systems do not coincide. B is a random set since it depends on the transition times. Now we would not expect agreement at all times, since after the exit of a particle the exclusion system must rearrange itself before it will again agree with the toy configuration; we can thus at most hope that B have small Lebesgue measure $|B|$. Moreover, we cannot rule out completely the possibility that $|B|$ will be large, since if there are many exits in the exclusion process before the necessary rearrangements occur, the eventual configuration may not agree with that of the toy model. These considerations motivate the following notion of near-agreement: for $\epsilon > 0$ we say that the two systems are ϵ-*close* if the probability that $|B| \geqslant \epsilon$ is less than ϵ.

Now we can state the main result of this appendix. We consider a limiting process in which β approaches 0 and admit the possibility that $T = T(\beta)$ and $N = N(\beta)$ increase during the limiting process; in particular, they may approach infinity, but not too rapidly.

Theorem. Suppose that $\eta(0) = \Phi(X(0))$ and that $T(\beta)$ and $N(\beta)$ are monotonically increasing functions which satisfy $\lim_{\beta \to 0} \beta NT = \lim_{\beta \to 0} \beta T^2 = 0$. Then for any $\epsilon > 0$ the two systems are ϵ-close for sufficiently small β.

Proof. For uniformity of notation it is convenient to assume that $T(\beta) \to \infty$ as $\beta \to 0$. This entails no loss of generality, since we may if necessary replace $T(\beta)$ by $T'(\beta) > T(\beta)$ with $T'(\beta) \to \infty$ and with $T'(\beta)$ still satisfying the given hypotheses (this is the only place that monotonicity is used), and observe that if the two systems are ϵ-close for $T'(\beta)$ then they are also ϵ-close for $T(\beta)$.

Let $t_1, t_2, \ldots$ be the exit times for the exclusion process, in increasing order, and let M be the number of these which occur between 0 and $T/(2\beta)$. M is a random variable having Poisson distribution with mean and hence variance T, and Chebyshev's inequality implies that the probability that $M \leqslant 2T$ is at least $1 - T^{-1}$ and hence goes to one as $\beta \to 0$. Let us denote by E_1 the event that $M \leqslant 2T$.

Now let r denote the minimum of the rates 1, q, and α, and define

$$a = a(\beta) = r^{-1}(4N + \beta^{-1/2}). \tag{A.1}$$

Because $1 \ll a \ll \beta^{-1}$, a time interval of length a is small on the τ scale but large on the t scale. It is during such an interval that agreement between the two processes will be restored after a particle exit—that is, during an interval $[t_i, t_i + a]$. Restoration is simplest to analyse if no additional exit occurs during the interval; note that the probability that the next exit time t_{i+1} satisfies $t_{i+1} < t_i + a$ is $1 - e^{-\beta a} \leqslant \beta a$. Thus the probability of the event that $t_{i+1} \geqslant t_i + a$ for all i satisfying $1 \leqslant i \leqslant 2T$ is at least $1 - 2T\beta a$, which by hypothesis goes to one as $\beta \to 0$. Let us call this event E_2. When both E_1 and E_2 occur, all the relevant exit times differ by at least a.

Suppose now that the two systems agree at the exit time t_i, i.e. $\eta(t_i) = \Phi(X(\tau_i))$, where $\tau_i = 2\beta t_i$. If an exit in fact occurs in the exclusion process at time t_i—for a left exit, in cases II and III of table A1—then agreement will not hold immediately thereafter. For each possible $X(\tau_i) \equiv (j, k)$, however, and knowing whether t_i is a left or right exit time, we may identify a (j, k)-*restoration event*: one or more sequences of entrance and exchange times whose occurrence during the interval $(t_i, t_i + a)$ will, if no exit time lies in this interval, guarantee the restoration of agreement by time $t_i + a$. We will describe these restoration events below and prove that the probability that a (j, k)-restoration event does not occur in a time interval $(t, t + a)$ is, for small β, uniformly bounded by $2(4N\beta + \beta^{1/2})$. Thus the probability of the event that a $X(\tau_i)$-restoration event occurs in the interval $(t_i, t_i + a)$ for all i satisfying $1 \leqslant i \leqslant 2T$ is at least $1 - 4T(N\beta + \beta^{1/2})$, which by hypothesis goes to one as $\beta \to 0$. Let us call this event E_3.

We can now complete the proof. Take β so small that the probability that all of E_1, E_2, and E_3 occur is greater than $1 - \epsilon$, and consider the evolution of the system when these three events do occur. Since the two systems agree at time zero they must agree until the first exit time t_1. Since no exit, and an $X(\tau_1)$-restoration event, occur during the interval $(t_1, t_1 + a)$, agreement will be restored by time $t_1 + a$, and will persist until time t_2. Continuing in this way, we see that agreement will hold for all t between 0 and $T/(2\beta)$ except for at most $2T$ time intervals of length a. Thus the total time of disagreement is at most $4\beta a T$ on the τ time scale. Finally, we take β so small that $4\beta a T < \epsilon$.

It remains to consider the restoration events and estimate their probabilities. In the calculation we will assume that all entrance and exchange processes occur at rate r, since in so doing we can only underestimate the probability of restoration events. Again let $t_i = \tau_i/(2\beta)$ be an exit time—to be definite, a left exit time—and set $X(\tau_i) = (j, k)$. We divide the discussion according to the cases of table A1.

Case III: $j \geqslant 2$. Agreement will certainly be restored if two sequences of transition times, corresponding to cases III.1 and III.2, both occur in the interval: (1) a left entrance

time, then $+- \rightarrow -+$ exchange times for bonds $\langle 1, 2 \rangle, \dots, \langle j - 1, j \rangle$ in succession, then $+0 \rightarrow 0+$ exchange times for bonds $\langle j, j+1 \rangle, \dots, \langle N - k - 1, N - k \rangle$, and (2) $0- \rightarrow -0$ exchange times for bonds $\langle 1, 2 \rangle, \dots, \langle j - 1, j \rangle$ in succession, followed if $k = 0$ (and hence $j = N$) by a right entrance time. Each of these sequences requires at most $N + 1$ transition times and hence has probability of non-occurrence at most the probability that $U < N + 1$, where U is a Poisson random variable with expectation (and hence variance) ra; by Chebyshev's inequality,

$$\text{Prob}[U < N + 1] \leqslant \frac{ra}{(ra - (N + 1))^2} \leqslant \frac{4N + \beta^{-1/2}}{\beta^{-1}} = 4N\beta + \beta^{1/2}. \tag{A.2}$$

Thus the event that one or the other sequence does not occur has probability at most $2(4N\beta + \beta^{1/2})$.

Case II: $j = 1$. Agreement will be restored when the system fills with $N - k +$ particles. It is easy to enumerate the entrances and exchanges required, but these may occur in many different orders, so that a direct computation of the time needed for restoration appears difficult. We adopt an alternative approach by comparing the model to a well understood system: a totally asymmetric one-species exclusion process on an infinite lattice, in which particles interchange with holes to their right at rate r. We suppose that sites in the one-species system are independently occupied or empty with probability $\frac{1}{2}$—this is a steady state—and couple the two systems by assuming that particle–hole exchanges on bonds $\langle 1, 2 \rangle, \dots, \langle N - 1, N \rangle$ can occur in the one-species model only at $+0 \rightarrow 0+$ exchange times for the same bonds in the two-species model, and that a $+0 \rightarrow 0+$ exchange on bond $\langle 0, 1 \rangle$ in the one-species model can occur only at a left entrance time for the two-species model. If we let $J(u)$ denote the number of particles which actually enter on the left in the two-species model during the time interval $(t_i, t_i + u)$, and $J'(u)$ the number of exchanges which actually take place on bond $\langle 0, 1 \rangle$ in the one-species model during this same time interval, then it is easy to verify that $J'(u) \leqslant J(u)$ as long as $J(u) < N - k$. From this it follows that the probability that $J(a) < N - k$, i.e. that restoration does not take place during the time interval $(t_i, t_i + a)$, is at most the probability that $J'(a) < N - k$. But it is shown in [17] that $J'(a)$ has expectation $ra/4$ and variance $V(a)$ satisfying $\lim_{a \to \infty} V(a)/ra = 0$; thus again by Chebyshev's inequality,

$$\text{Prob}[J(a) < N - k] \leqslant \frac{V(a)}{(ra/4 - (N - k))^2} \leqslant \frac{ra}{\beta^{-1}} = 4N\beta + \beta^{1/2} \tag{A.3}$$

for β sufficiently small that $V(a) \leqslant ra$.

Appendix B

This appendix is devoted to an algorithm which allows for a fast recursive solution of the difference equations (4.10).

• We start with the $v_{j,k}$ for the sake of simplicity. Setting $j + k = n$, and introducing the notation $v_{j,k} = v_j^{(n)}$, the recursion relation (4.10b) reads

$$D v_j^{(n)} = F_j^{(n-1)} \tag{B.1}$$

with the boundary conditions $v_0^{(n)} = 1$, $v_n^{(n)} = 0$. In equation (B.1) D is the linear operator defined as

$$D v_j^{(n)} = v_j^{(n)} - b(v_{j-1}^{(n)} + v_{j+1}^{(n)}) \tag{B.2}$$

and the source term reads

$$F_j^{(n-1)} = a(v_j^{(n-1)} + v_{j-1}^{(n-1)}). \tag{B.3}$$

Equation (B.1) is a linear equation for the $v_j^{(n)}$, with a source term $F_j^{(n-1)}$ involving the $v_k^{(n-1)}$, whence the possibility of a recursive solution.

The solution to equation (B.1) is the superposition

$$v_j^{(n)} = (v_H)_j^{(n)} + (v_I)_j^{(n)} \tag{B.4}$$

of the solutions of the following two equations

$$D(v_H)_j^{(n)} = 0 \qquad \text{with } (v_H)_0^{(n)} = 1 \qquad (v_H)_n^{(n)} = 0 \tag{B.5a}$$

$$D(v_I)_j^{(n)} = F_j^{(n-1)} \qquad \text{with } (v_I)_0^{(n)} = 0 \qquad (v_I)_n^{(n)} = 0. \tag{B.5b}$$

(with H for homogeneous, I for inhomogeneous).

The solution of the homogeneous equation (B.5a) goes as follows. Searching a solution of the form z^j, we find $z = \exp(\pm\sigma)$, with the notation (6.12). Hence, imposing the boundary conditions, we obtain

$$(v_H)_j^{(n)} = \frac{U_{n-j}}{U_n} \tag{B.6}$$

where

$$U_n(\zeta) = \frac{\sinh(n\sigma)}{\sinh\sigma} \tag{B.7}$$

is the Chebyshev polynomial of the second kind and of order n in the variable

$$\zeta = 2\cosh\sigma = 1/b. \tag{B.8}$$

These polynomials obey the recursion relation

$$U_{n+1} + U_{n-1} = \zeta U_n \tag{B.9}$$

with $U_0(\zeta) = 0$, $U_1(\zeta) = 1$, so that $U_2(\zeta) = \zeta$, $U_3(\zeta) = \zeta^2 - 1$, etc.

The solution of the inhomogeneous equation (B.5b) reads

$$(v_I)_j^{(n)} = \sum_{\ell=1}^{n-1} G_{j,\ell}^{(n)} F_\ell^{(n-1)} \tag{B.10}$$

where $G_{j,\ell}^{(n)}$ is the Green function of D, defined by

$$DG_{j,\ell}^{(n)} = \delta_{j,\ell} \qquad \text{with } G_{0,\ell}^{(n)} = G_{n,\ell}^{(n)} = 0. \tag{B.11}$$

The Green function can be obtained explicitly as a superposition of plane waves $\exp(\pm j\sigma)$ in each of the sectors defined by the inequalities $0 \leqslant j \leqslant \ell$ and $\ell \leqslant j \leqslant n$. We thus obtain the simple expression

$$G_{j,\ell} = \frac{2\cosh\sigma}{U_n} U_{\inf(j,\ell)} U_{n-\sup(j,\ell)} \tag{B.12}$$

where inf(j, ℓ) denotes the smaller of the two integers, i.e. ℓ if $\ell < j$ and j otherwise, whereas sup(j, ℓ) denotes the larger, i.e. j if $\ell < j$ and ℓ otherwise.

The full recursive solution to equation (B.1) thus reads

$$v_j^{(n)} = \frac{U_{n-j}}{U_n} + \frac{\cosh\sigma - 1}{U_n} \sum_{\ell=1}^{n-1} U_{\inf(j,\ell)} U_{n-\sup(j,\ell)}(v_\ell^{(n-1)} + v_{\ell-1}^{(n-1)}). \qquad (B.13)$$

• Likewise one may write a recursion relation for the $u_j^{(n)}$, namely

$$u_j^{(n)} = \frac{2\cosh\sigma}{U_n} \sum_{\ell=1}^{n-1} U_{\inf(j,\ell)} U_{n-\sup(j,\ell)} + \frac{\cosh\sigma - 1}{U_n} \sum_{\ell=1}^{n-1} U_{\inf(j,\ell)} U_{n-\sup(j,\ell)}(u_\ell^{(n-1)} + u_{\ell-1}^{(n-1)}). \qquad (B.14)$$

One may deduce from the above analysis that the $v_j^{(n)}$ and $u_j^{(n)}$ are rational expressions in a, or equivalently in b. Their denominators are products of the Chebyshev polynomials up to U_n. As a consequence, as n gets large, the poles of the $v_j^{(n)}$ and $u_j^{(n)}$ become everywhere dense in the range $-1 < 1/(2b) < 1$. In terms of b, the poles cover the whole real axis except the interval $[-\frac{1}{2}, \frac{1}{2}]$. In terms of a, the poles cover the whole real axis except the interval $[0, 1]$.

The recursive solutions derived above provide a fast and efficient way of computing the $u_{j,k}$ and the $v_{j,k}$ numerically.

• The stationary probabilities $p_j^{(\nu)}$ can be determined by means of a similar recursive scheme. First, equation (6.1) is a closed-form equation for the $p_j^{(0)}$, whose solution reads

$$p_j^{(0)} = p_0^{(0)} \frac{U_j + U_{N-j}}{U_N}. \qquad (B.15)$$

Second, the structure of equation (6.1) for $\nu \geqslant 1$ is very similar to that of equation (B.1). We thus obtain the following recursion relation for the $p_j^{(\nu)}$

$$p_j^{(\nu)} = \frac{\cosh\sigma - 1}{U_n} \sum_{\ell=1}^{N-\nu-1} U_{\inf(j,\ell)} U_{N-\nu-\sup(j,\ell)}(p_\ell^{(\nu-1)} + p_{\ell-1}^{(\nu-1)}). \qquad (B.16)$$

Third, the first probability $p_0^{(0)}$ is then determined by means of the normalization condition (4.5).

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